The Burns Guide to
Uncertainties and Error Propagation

1. How to Round Answers Correctly

This method is also known as *unbiased rounding* or as *statistician's rounding* or as *bankers' rounding*. It is identical to the common method of rounding except when the digit(s) following the rounding digit start with a five and have no non-zero digits after it. The new algorithm is:

- Decide which is the last digit to keep.
- Increase it by 1 if the next digit is 6 or more, or a 5 followed by one or more non-zero digits.
- Leave it the same if the next digit is 4 or less
- Otherwise, all that follows the last digit is a 5 and possibly trailing zeroes; then change the last digit to the nearest *even* digit. That is, increase the rounded digit if it is currently odd; leave it if it is already even

Examples

a) round 7298 to the nearest hundreds place
b) round 345 to the tens place
c) round 33299 to the nearest thousands place
d) round 3.0447 to the hundredths place
e) round –2.1350 to the hundredths place
f) round 3.045 to the hundredths place.
g) round 3.04501 to the hundreths place rounded to hundredths is 3.05 (because the next digit is 5, but it is followed by non-zero digits)

Solution:

a) 7300 (the 9 in the tens place indicated you should round up)
b) 340 (the next difit is a five, therefore since 4 is even, leave it alone)
c) 33000 (the 2 in the hundreds place indicates you don't round up)
d) 3.04 (the next digit is a 4 therefore you don't round up)
e) –2.14 (the next digit is a five, therefore since 3 is odd increase from 3 to 4)
f) 3.04 (because the next digit is 5, and the hundredths digit (4) is even)
g) 3.05 (because the next digit is 5, but it is followed by non-zero digits

Recall: The uncertainty should be rounded off to one or two significant figures. If the leading figure in the uncertainty is a 1, we use two significant figures; otherwise we use one significant figure. Then the answer should be rounded to match.

Example

Round z=11.0249 cm and $\Delta z = 0.154 cm$

Solution:

Since Δz begins with a 1, we round Δz to two significant digits: $\Delta z = 0.15$ cm.

Thus we have to round z to the same number of decimal places: $z = \left(11.02 \pm 0.15\right) cm$

Note:

When the answer is given in scientific notation, the uncertainty should be given in scientific notation with the **same power of ten**. Thus, if $z = 1.67 \times 10^6$ s and $\Delta z = 3 \times 10^4$ s, we should write our answer as $z = (1.67 \pm 0.03) \times 10^6$ s.

Examples

Express the following results in proper rounded form, x ± Δx.

(i) m = 24.34506 grams, Δm = 0.04252 grams.
(ii) t = 0.03346 sec, Δt = 1.623 x 10^{-3}sec.
(iii) M = 8.35 x 10^{22} kg ΔM = 2.6 x 10^{20} kg.
(iv) m = 9.11 x 10^{-33} kg Δm = 3.2345 x 10^{-33} kg

Solution:

(i) (24.35 ± 0.04) gram

(ii) $(3.35 \pm 0.16) \times 10^{-2}$ sec

(iii) $(8.35 \pm 0.03) \times 10^{22}$ kg

(iv) $(9 \pm 3) \times 10^{-33}$ kg

2. Significant Figures

The rules for propagation of errors hold true for cases when we are in the lab, but doing propagation of errors is time consuming. The rules for significant figures allow a much quicker method to get results that are approximately correct even when we have no uncertainty values.

A significant figure is any digit 1 to 9 and any zero which is not a place holder. Thus, in 2.450 there are 4 significant figures since the zero is not needed to make sense of the number. In a number like 0.00240 there are 3 significant figures --the first three zeros are just place holders. However the number 2450 is ambiguous. You cannot tell if there are 3 significant figures --the 0 is only used to hold the units place --or if there are 4 significant figures and the zero in the units place was actually measured to be zero.

How do we resolve ambiguities that arise with zeros when we need to use zero as a place holder as well as a significant figure? Suppose we measure a length to three significant figures as 9000 cm. Written this way we cannot tell if there are 1, 2, 3, or 4 significant figures. To make the number of significant figures apparent we use scientific notation, 9 x 10^3 cm (which has one significant figure), or 9.00 x 10^3 cm (which has three significant figures), or whatever is correct under the circumstances.

We start then with values each with their own number of significant figures and compute a new quantity. How many significant figures should be in the final answer? In doing running computations we maintain numbers to many figures, but we must report the answer only to the proper number of significant figures.

Addition and Subtraction

- For addition and subtraction, look at the decimal portion (i.e., to the right of the decimal point) of the numbers ONLY. Here is what to do:
- Count the number of significant figures in the decimal portion of each number in the problem. (The digits to the left of the decimal place are not used to determine the number of decimal places in the final answer.)
- Add or subtract in the normal fashion.
- Round the answer to the LEAST number of places in the decimal portion of any number in the problem.

Example

Suppose one object is measured to have a mass of 9.8 gm and a second object is measured on a different balance to have a mass of 0.4256 gm. What is the total mass? We write the numbers with question marks at places where we lack information. Thus 9.8???? gm and 0.4256? gm. Adding them with the decimal points lined up we see

9.8 ????
0.4256?
10.2 ???? = 10.2 gm.

As you can see the least number of places after the decimal portion is 1 (in 9.8), therefore our answer must be written to only 1 decimal place.

Multiplication

The following rule applies for multiplication and division:

The LEAST number of significant figures in any number of the problem determines the number of significant figures in the answer.

This means you MUST know how to recognize significant figures in order to use this rule.

Example

Evaluate 2.5 x 3.42

Solution:

Now 8.55 is the answer from a calculator, but 2.5 has **2** significant figures and 3.42 has **3** significant figures, therefore our answer must be correct to **only 2 significant figures**. That is

2.5 x 3.42 = 8.6

It is important to keep these concepts in mind as you use calculators with 8 digit or more displays. If you are to avoid mistakes in your answers and wish to avoid the wrath of physics instructors everywhere. A good procedure to use is to use all digits (significant or not) throughout your calculations (*some professors will allow only 1 more digit than is significant in intermediate steps, follow their instructions*), and only round off the answers at the last step to appropriate number of significant digits.

Examples

a) $\left(3.4617\times10^{7}\right)\div\left(5.61\times10^{-4}\right)$

b) $\dfrac{\left(9.714\times10^{5}\right)\left(2.1482\times10^{-9}\right)}{\left(4.1212\right)\left(3.7792\times10^{-5}\right)}$

c) $\dfrac{\left(4.7620\times10^{-15}\right)}{\left(3.8529\times10^{12}\right)\left(2.813\times10^{-7}\right)\left(9.50\right)}$

Solutions:

a) The calculator shows 6.1706 x 10^{10} which then rounds to 6.17 x 10^{10}
b) The calculator shows 1.3398 x 10^{1} which then rounds to 13.40
c) The calculator shows 4.625×10^{-22}, which then rounds to 4.625×10^{-22}

3. Random Errors versus Systematic Errors

No matter how careful we are and no matter how expensive our equipment is, no measurement made is ever exact. The **accuracy** (correctness) and **precision** (number of significant figures) of any measurement is always limited by a variety of factors:

- the skill of the observer (that is you or your lab partner)
- the calibration the measuring equipment is capable of
- the environment in which the experiment is performed.

When we perform an experiment we are either trying to establish the best values for certain quantities, or trying to validate a theory. When we obtain experimental values we need to ensure that we include a **range of possible true values** based on our limited number of measurements.

We use a variety of terms such as **uncertainty, error, variance, or deviation** to represent the disparity in measured data. Two types of errors are possible:

Systematic errors are reproducible inaccuracies that are consistently in the same direction. Systematic errors are often due to a problem that persists throughout the entire experiment and is usually the result of a mis-calibrated device, or a measuring technique that always makes the measured value larger (or smaller) than the "true" value. The electronic scale you use reads 0.05 g too high for all your mass measurements (because it was improperly zeroed at the beginning of your experiment). Systematic errors are difficult to detect and cannot be analyzed statistically, because all of the data is off in the same direction (either to high or too low). Spotting and correcting for systematic error takes a lot of care. Yet, careful design of an experiment and experimental procedure will allow us to eliminate or to correct for systematic errors.

Even if we eliminated then systematic errors there will remain a second type of variation in measured values of a single quantity. These remaining deviations are classed as **random errors.** Random errors are statistical fluctuations (in either direction) in the measured data due to the precision limitations of the measurement device. Random errors usually result from the experimenter's inability to take the same measurement in exactly the same way to get exact the same value. You measure the mass of a wooden block four times using the same balance and get slightly different values: 57.46 g, 57.48 g, 57.45g, and 57.47g.

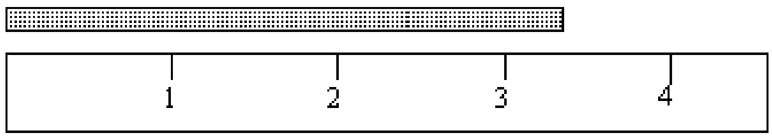
4. Determining random errors.

How can we estimate the uncertainty of a measured quantity? Several approaches can be used, depending on the application.

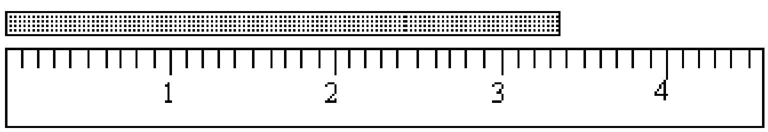
(a) Least Count and Measurement Estimate

The **least count** is the smallest division that is marked on the instrument. Thus a meter stick will have a least count of 1.0 mm, a balance scale might have a least count of 1 g.

On the ruler below, the least count is 1 cm, and we could estimate the gray bar to be about 3 1/4 cm



On the ruler below the least count is 0.1 cm, and we could estimate the length of the gray bar to be 3.35 cm



The **Measurement Estimate** is the precision to which a measuring device can be read, and is always equal to or smaller than the least count. The measurement estimate is generally taken to be the least count or some fraction (1/2, 1/3, 1/4) of the least count. The most common question that students have is what fraction of the least count should I pick? There is no exact answer to this question, but instead you must be guided by common sense. The larger the spacing between the division marks, the more comfortable you will be in estimating a fraction of the least count (1/4 or 1/5 or even 1/10). If the spacing between the division marks is small, you may only be able to estimate to the least count or 1/2 of the least count.

(b) The Uncertainty Estimate

The inability to read instrumentation to an exact value is an example of uncertainty. Often we encounter other uncertainties larger than the least count or measurement errors. The underlying questions here are: How do we determine this uncertainty? How do we minimize these uncertainties? How do we perform calculations with uncertainties? How do uncertainties propagate through out our experiment?

c) Uncertainty by Repeated Measurements

If you are unsure of the uncertainty, or the result has no uncertainty attached to it, the preferred method of determining the uncertainty for that piece of data (yes there will always be an uncertainty) is to repeat the experiment many times and then use statistical methods to determine the uncertainty.

Procedure :

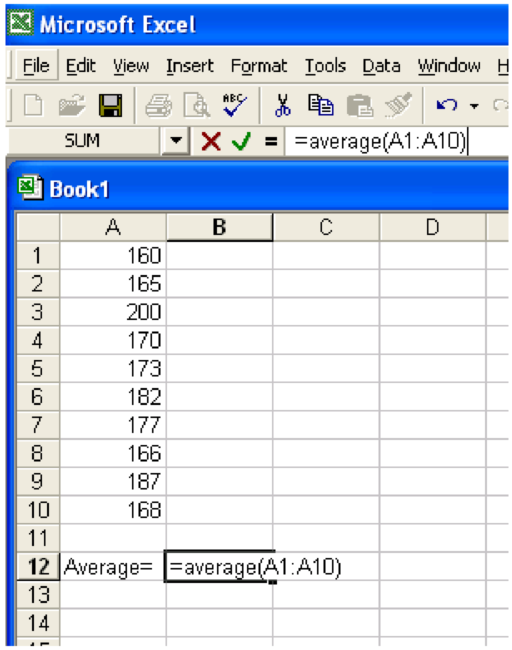
Step 1: Calculate the average value of the data
Step 2: Calculate the standard deviation of the data
Step 3: Calculate the standard error of the mean

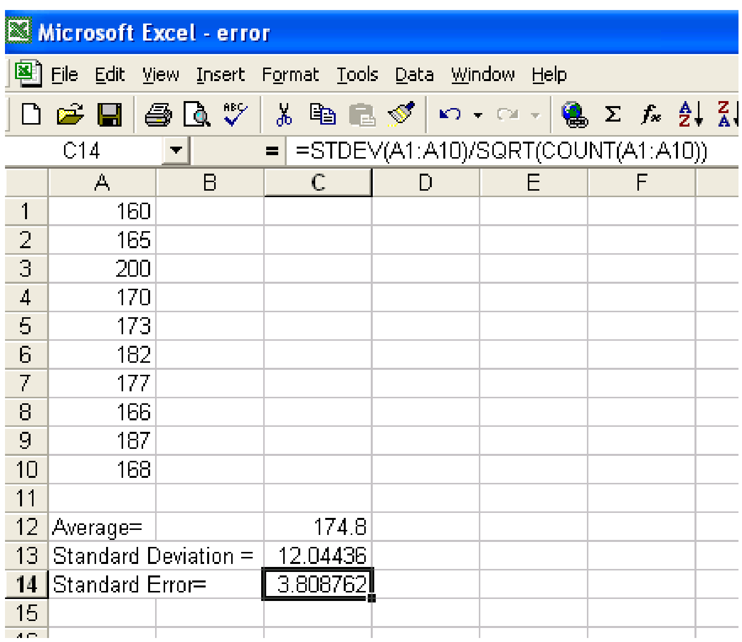
$$\frac{\text{Standard Deviation}}{\sqrt{n}}$$

Use excel to assist you in these calculations

Example: Suppose you collect the following ten data points:
{160, 165, 200, 170, 173, 182, 177, 166, 187, 168}

Step 1 : Using excel: =average(A1..A10)
Step 2 : Using excel: =stdev(A1..A10)
Step 3 : Using excel: =stdev(A1..A10)/SQRT(Count(A1..A10))





Therefore the average value is 174.8 ± 3.8 which is better stated as 175 ± 4.

Typically we round uncertainty to 1 significant digit, but some institutions (or teachers) prefer 2 significant digits.

Follow your instructor's instructions on whether to use average or standard deviation in your reports and to the number of uncertainty significant digits.

Example

Find the average value for the length of an object with error, when 10 measurements have been taken: {87cm, 88cm, 88.5cm, 88cm, 86.8cm, 87.5cm, 89.2cm, 87.4cm, 88.4cm, 87.4cm}.

Solution: **Average value:** 87.87
Standard deviation: 0.74057
Standard error of mean: 0.234189

Therefore the average value plus uncertainty is (87.9 ± 0.2)cm

You didn't forget units, Did you?

(d) Conflicts in the above

In some situations we will get a measurement error, an estimated uncertainty, and an average error and we notice that there are different error values for each of these. In these situations, you will be pessimistic and **take the largest of the three values as our uncertainty**.

A very common question students ask is : "Should I include every uncertainty technique in my answer?" The answer is simply, No. This will make life simple (and simple is good), but physicists do not expect that every single measurement taken in an experiment falls within the uncertainty interval that you specify. Unusual things happen, and (particularly if you take a lot of data) some of your data will probably reflect unusual situations. There shouldn't be too many of these unusual situations (or they wouldn't be unusual, now would they?), but unusual things are bound to happen. The uncertainty interval should reflect the range in which we can reasonably expect a "reasonable" value to fall.

For example we might measure a mass of a substance measurement error of 0.02 grams and an estimated uncertainty of 0.1 gram. We will use 0.1 gram as our uncertainty.

The proper way to write the answer is:

- Choose the largest of (a) measurement error, (b) estimated uncertainty, and (c) average or standard deviation.
- Round off the uncertainty to 1 significant figure.
- Round off the answer so it has the same number of digits before or after the decimal point as the uncertainty.
- Put the answer and its uncertainty in parentheses, then put the power of 10 and unit outside the parentheses.

Example

I have a measured quantity of **15.12394 g** and a calculated error of **0.06211 g**. how should I write the mass with its uncertainty?

Solution:

(15.12±0.06) g

5. Relative and Absolute Errors

The quantity Δz is called the **absolute error** while $\frac{\Delta z}{z}$ is called the **relative error** or **fractional uncertainty**.

Percentage error is the fractional error multiplied by 100%. In practice, either the percentage error or the absolute error may be provided.

Example: You are given a resistor with a resistance of 1500 ohms and a tolerance of 6%. What is the absolute error, relative error in the resistance?

Solution: The absolute error is 6% of 1500 ohms = 90 ohms.

The relative error is $\frac{90}{1500} = 6\%$

6. Percentage Error and Percent difference

The percent error can be found only if it is possible to compare an experimental value with that of the most commonly accepted value. The formula is:

$$\text{\% Error} = \frac{\text{absoluite error}}{\text{accepted value}} \times 100\% = \frac{\text{measured value - accepted value}}{\text{accepted value}} \times 100\%$$

Example: What is the percent error on the measurement of $9.5\frac{m}{s^2}$ for the acceleration due to gravity?

Solution: % error $= \frac{9.5\frac{m}{s^2} - 9.8\frac{m}{s^2}}{9.8\frac{m}{s^2}} \times 100\%$

$= -3.1\%$

The negative sign indicates that the measurement value was less than the accepted value.

The percentage difference: $= \frac{\text{absolute error}}{\text{average measurement}} \times 100\% = \frac{\text{difference in measurement}}{\text{average measurement}} \times 100\%$

Example: Two measurements of the acceleration due to gravity are $9.4\frac{m}{s^2}$ and $9.7\frac{m}{s^2}$. Determine the percentage difference.

Solution:

$$\text{percentage difference} = \frac{9.7\frac{m}{s^2} - 9.4\frac{m}{s^2}}{\left(\frac{9.7\frac{m}{s^2} + 9.4\frac{m}{s^2}}{2}\right)} \times 100\%$$

$$= \frac{0.3\frac{m}{s^2}}{9.55\frac{m}{s^2}} \times 100\%$$

$$= 3.1\%$$

7. Propagation of Errors, Basic Rules

Suppose we are given two measured quantities x and y and that these quantities have uncertainties, Δx and Δy, as determined above: we would report x ± Δx, and y ± Δy. Now suppose that from the measured quantities x and y a new quantity, z, is calculated. What is the uncertainty, Δz, in z?

There are a variety of techniques for determining the uncertainty Δz, in z . Some techniques are acceptable for high school, but not acceptable for University or Business. I will demonstrate three methods (GOOD, BETTER, BEST). BEST is usually always the method that you will use at university (so practice learning it), while the GOOD and BETTER methods will be acceptable at the High School Level. The guiding principle in all cases is to consider the most pessimistic situation.

The examples included in this section also involve the proper rounding of answers, which is covered in more detail in Section 6. The examples use the propagation of errors using average deviations on the left and standard deviations on the right.

Addition and Subtraction: z = x + y or z = x - y

Derivation: We will assume that the uncertainties are arranged so as to make z as far from its true value as possible.

Average deviations Δz = |Δx| + |Δy| in both cases

With more than two numbers added or subtracted we continue to add the uncertainties

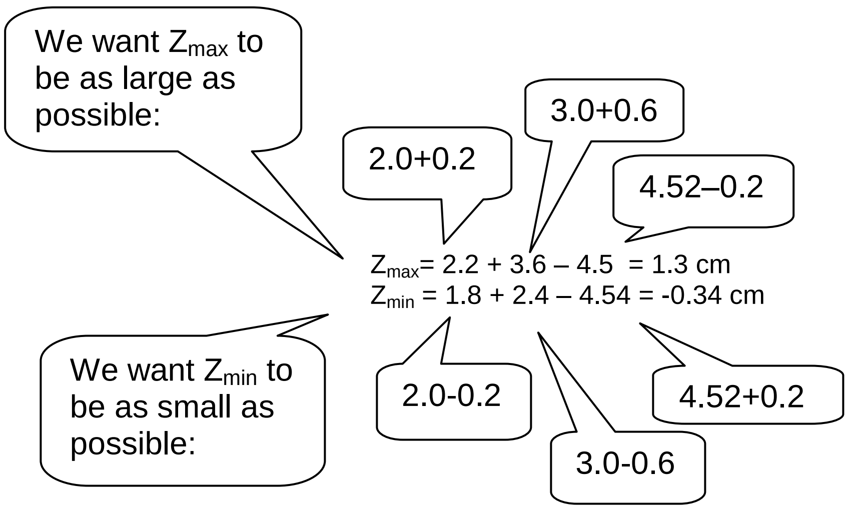
GOOD	BETTER	BEST
Δz is the largest value from z	$\Delta z = \lvert\Delta x\rvert + \lvert\Delta y\rvert + \dots$	$\Delta z = \sqrt{(\Delta x)^2 + (\Delta y)^2 + \dots}$

Example: w = (4.52 ± 0.02) cm, x = (2.0 ± 0.2) cm, y = (3.0 ± 0.6) cm

Determine z= x + y - w with error

Solution: z = x + y - w = 2.0 + 3.0 - 4.52 = 0.5 cm

<u>GOOD:</u> We must determine the largest possible error for z



The distance z is from z_{max} is 0.8 cm
The distance z is from z_{min} is 0.84 cm

We choose the largest of these distances (0.84 cm) and round to 1 digit : 0.8 cm

Thus z = = (0.5 ± 0.8) cm

<u>BETTER</u>

Δz = Δx + Δy + Δw = 0.2 + 0.6 + 0.02 = 0.82

rounding to 0.8 cm

Thus z = (0.5 ± 0.8) cm

<u>BEST</u>

$$\begin{aligned}\Delta z &= \sqrt{(0.02)^2 + (0.2)^2 + (0.6)^2} \\ &= \sqrt{0.4004} \\ &= 0.63277\end{aligned}$$

So z = (0.5 ± 0.6) cm

Notice that we rounded the uncertainty to one significant figure and rounded the answer to match

Multiplication by a Constant

Multiply the uncertainty by the same exact number.

Example: y = (3.0 ± 0.2) cm. Find C = 6 y

Solution: ΔC = 6(Δy) = 1.2 cm

C = (18.0 ± 1.3) cm or C = (18 ± 1) cm

We can round the uncertainty to two figures since it starts with a 1, and round the answer to match.

Example: x = (2.0 ± 0.2) cm, y = (3.0 ± 0.6) cm. Find z = 3x - 2y.

Solution: z = 3x - 2y = 3(2.0) - 2(3.0) = 0.0 cm

GOOD

Z_{max} =3(2.2)-2(2.4) = 1.8
Z_{min} =3(1.8)-3(3.6) =-5.4

Therefore z=(0.0 ± 5.0) cm

BETTER

Δz = 3 Δx + 2 Δy = 0.6 + 1.2 = 1.8 cm

So z = (0.0 ± 1.8) cm.or z = (0.0 ± 2.0) cm

BEST

$$\begin{aligned}\Delta z &= \sqrt{3(\Delta x)^2 + 2(\Delta y)^2} \\ &= \sqrt{3(0.2)^2 + 2(0.6)^2} \\ &= \sqrt{0.84} \\ &= 0.9165\end{aligned}$$

So z = (0.0 ± 0.9) cm.

Multiplication and Division: z = x y or z = x/y

Derivation: We can derive the relation for multiplication easily. Take the largest values for x and y, that is

$z + \Delta z = (x + \Delta x)(y + \Delta y) = xy + x\,\Delta y + y\,\Delta x + \Delta x\,\Delta y$

Usually $\Delta x \ll x$ and $\Delta y \ll y$ so that the last term is much smaller than the other terms and can be neglected. Since z = xy,

$\Delta z = y\,\Delta x + x\,\Delta y$

which we write more compactly by forming the relative error, that is the ratio of $\Delta z/z$, namely

$$\frac{\Delta z}{z} = \frac{\Delta x}{x} + \frac{\Delta y}{y} + \dots$$

The same rule holds for multiplication, division, or combinations, namely add all the relative errors to get the relative error in the result.

From now on, the **GOOD** method technique is always the same (no formulas to remember) so we will omit it from the formula section.

BETTER	BEST
$\Delta z = z\left(\frac{\Delta x}{x} + \frac{\Delta y}{y} + \dots\right)$	$\Delta z = z\sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2 + \dots}$

Example: w = (5.42 ± 0.03) cm, x = (3.0 ± 0.2) cm. Find z = w x.

Solution: z = w x = (5.42) (3.0) = 16.26 cm^2

<u>GOOD</u>

z_{max} = (5.45)(3.2)=17.44
z_{min} = (5.39)(2.8)=15.092

$z-z_{max}$=16.26-17.44=-1.18
$z-z_{min}$ =16.26-15.092=1.168

We will uses $\Delta z = 1.2$

Therefore z=(16.3 ± 1.2) cm^2 or z=(16 ± 1) cm^2 is also acceptable

BETTER

$$\Delta z = z\left(\frac{\Delta w}{w}+\frac{\Delta x}{x}\right)$$
$$=16.26\left(\frac{0.03}{5.42}+\frac{0.2}{3.0}\right)$$
$$=1.174$$

Therefore z=(16.3 ± 1.2) cm^2 or z=(16 ± 1) cm^2 is also acceptable

BEST

$$\Delta z = z\sqrt{\left(\frac{\Delta w}{w}\right)^2+\left(\frac{\Delta x}{x}\right)^2}$$
$$=16.26\sqrt{\left(\frac{0.03}{5.42}\right)^2+\left(\frac{0.2}{3.0}\right)^2}$$
$$=1.0877$$

Therefore z=(16.3 ± 1.1) cm^2 or z=(16 ± 1) cm^2 is also acceptable.

Example: w = (5.42 ± 0.03) cm, x = (3.0 ± 0.2) cm. Find $z=\frac{w}{x}$.

Solution: $z=\frac{w}{x}=\frac{5.42}{3.0}=1.8067$ cm^2

GOOD

z_{max} = (5.45)/(2.8)=1.9464
z_{min} = (5.39)(2.8)=1.6844

z-z_{max}=16.26-17.44= -0.1397
z-z_{min} =16.26-15.092=0.1223

We will uses $\Delta z = 0.1$

Therefore z=(1.8 ± 0.1) cm^2

BETTER

$$\Delta z = z\left(\frac{\Delta w}{w}+\frac{\Delta x}{x}\right)$$
$$=1.8067\left(\frac{0.03}{5.42}+\frac{0.2}{3.0}\right)$$
$$=0.1343$$

Therefore z=(1.8 ± 0.1) cm^2

BEST

$$\Delta z = z\sqrt{\left(\frac{\Delta w}{w}\right)^2 + \left(\frac{\Delta x}{x}\right)^2}$$

$$= 1.8607\sqrt{\left(\frac{0.03}{5.42}\right)^2 + \left(\frac{0.2}{3.0}\right)^2}$$

$$= 0.1245$$

Therefore z=(1.8 ± 0.1) cm^2 .

Products of powers: $z = x^m y^n$.

The results in this case are:

Better	Best
$\Delta z = z\left(\lvert m\rvert\frac{\Delta x}{x}+\lvert n\rvert\frac{\Delta y}{y}+...\right)$	$\Delta z = z\sqrt{\left(\frac{m\Delta x}{x}\right)^2+\left(\frac{n\Delta y}{y}\right)^2+...}$

Example: w = (5.42 ± 0.02) cm, x = (3.0 ± 0.2) cm^2, y = (2.0 ± 0.3) cm.

Find $z = \frac{y^2 w}{\sqrt{x}}$

Solution: $z = \frac{y^2 w}{\sqrt{x}} = \frac{(2.0cm)^2(5.42cm)}{\sqrt{3.0cm^2}} = 12.5169cm^2$

GOOD

z_{max} = $z = \frac{y^2 w}{\sqrt{x}} = \frac{(2.3cm)^2(5.44cm)}{\sqrt{2.8cm^2}} = 17.1979cm^2$

z_{min} = $z = \frac{y^2 w}{\sqrt{x}} = \frac{(1.7cm)^2(5.40cm)}{\sqrt{3.2cm^2}} = 8.7240cm^2$

$z-z_{max}=12.5169-17.1979=-4.681$
$z-z_{min}=12.5169-8.7240=3.7929$

We will uses $\Delta z = 5$

Therefore z=(13± 5) cm^2

BETTER

$$\Delta z = z\left(2\frac{\Delta y}{y}+\frac{\Delta w}{w}+0.5\frac{\Delta x}{x}\right)$$
$$=12.5169\left(2\frac{0.3}{2.0}+\frac{0.02}{5.42}+0.5\frac{0.2}{3.0}\right)$$
$$=4.2185$$

Therefore z=(13 ± 4) cm^2

BEST

$$\Delta z = z\sqrt{\left(2\frac{\Delta y}{y}\right)^2+\left(\frac{\Delta w}{w}\right)^2+\left(0.5\frac{\Delta x}{x}\right)^2}$$
$$=12.5169\sqrt{\left(2\frac{0.3}{2.0}\right)^2+\left(\frac{0.02}{5.42}\right)^2+\left(0.5\frac{0.2}{3.0}\right)^2}$$
$$=3.7785$$

Therefore z=(13 ± 4) cm^2

Mixtures of Multiplication, Division, Addition, Subtraction, and Powers.

If z is a function which involves several terms added or subtracted we must apply the above rules carefully. This is best explained by means of an example.

Example

Given: w = (5.42 ± 0.03) cm, x = (3.0 ± 0.2) cm, y = (4.0 ± 0.4) cm determine z when $z = wx + y^2$

Solution:

First determine the value of z

$$\begin{aligned} z &= wx + y^2 \\ &= (5.42\,cm)(3.0\,cm) + (4.0\,cm)^2 \\ &= 32.26\,cm^2 \end{aligned}$$

Now for the uncertainty value

GOOD

$$z_{\max} = (5.45)(3.2) + (4.4)^2 = 36.8$$
$$z_{\min} = (4.39)(2.8) + (3.6)^2 = 28.052$$

$$z - z_{\max} = 32.26 - 36.8 = -4.54$$
$$z - z_{\min} = 32.26 - 28.052 = 4.208$$

Therefore we will use $\Delta z = 5$

Thus $z = (32 \pm 5)\,cm^2$

BETTER

$$\begin{aligned} \Delta z &= |\Delta wx| + |\Delta y^2| \\ &= wx\left(\frac{\Delta w}{w} + \frac{\Delta x}{x}\right) + y^2\left(2\frac{\Delta y}{y}\right) \\ &= 16.26\left(\frac{0.03}{5.42} + \frac{0.2}{3.0}\right) + 16\left(2\frac{0.4}{4.0}\right) \\ &= 4.374 \end{aligned}$$

Therefore we have $z = (32 \pm 4)\,cm^2$

BEST

$$\Delta z = \sqrt{(\Delta wx)^2 + (\Delta y^2)}$$
$$= \sqrt{\left(wx\sqrt{\left(\frac{\Delta w}{w}\right)^2 + \left(\frac{\Delta x}{x}\right)^2}\right)^2 + \left(y^2\sqrt{\left(\frac{2\Delta y}{y}\right)^2}\right)^2}$$
$$= \sqrt{\left(16.26\sqrt{\left(\frac{0.03}{5.42}\right)^2 + \left(\frac{0.2}{3.0}\right)^2}\right)^2 + \left(16\sqrt{\left(2\frac{0.4}{4.0}\right)^2}\right)^2}$$
$$= 3.380$$

Therefore we have $z = (32 \pm 3)cm^2$

Other Functions: (Trigonometric, Logarithmic Functions)

For non-polynomial functions, such as trigonometric functions or logarithm functions, there are two methods that are acceptable. The first assumes you have no Calculus knowledge and is our usual **GOOD** method. The second technique assumes you are comfortable with Calculus, this technique was used to derive our **BEST** formulas in previous examples.

BEST If q=f(x_1, x_2, x_3, …x_n) then $\Delta q = \sqrt{\left(\frac{\partial q}{\partial x_1}\Delta x_1\right)^2 + \left(\frac{\partial q}{\partial x_2}\Delta x_2\right)^2 + \ldots + \left(\frac{\partial q}{\partial x_n}\Delta x_n\right)^2}$

Example

If $S = x\cos(\theta)$ with $x = (3.0 \pm 0.2)\,cm$ and $\theta = (47 \pm 2)^\circ$

Solution:

First determine the value of S

$$S = (3.0)\cos(47^\circ) = 2.046$$

GOOD

$$S_{\max} = (3.2)\cos(45^\circ) = 2.263$$
$$S_{\min} = (2.8)\cos(49^\circ) = 1.837$$

$$S - S_{\max} = 2.046 - 2.263 = -0.217$$
$$S - S_{\min} = 2.046 - 1.837 = 0.209$$

Therefore $S = (2.0 \pm 0.2)^\circ$

BEST

Note: All trigonometric functions must be in radians

$$\theta = (47 \pm 2)^\circ \Rightarrow (0.82030 \pm 0.03491)\ rad$$

$$\begin{aligned}\Delta S &= \sqrt{\left(\frac{\partial S}{\partial x}\Delta x\right)^2 + \left(\frac{\partial S}{\partial \theta}\Delta\theta\right)^2} \\ &= \sqrt{\left(\cos(\theta)\Delta x\right)^2 + \left(-x\sin(\theta)\Delta\theta\right)^2} \\ &= \sqrt{\left(\cos(0.82030)(0.2)\right)^2 + \left(-3.0\sin(0.82030)\cdot(0.03491)\right)^2} \\ &= 0.1999\end{aligned}$$

Therefore $S = (2.0 \pm 0.2)^\circ$

Example

The Atwood Machine consists of two masses M and m attached to the ends of a light, frictionless pulley. When the masses are released, the mass M is show to accelerate down with an acceleration: $a = g\frac{M-m}{M+m}$.
Suppose the M and m are measured as M=(100 ±1)g and m=(50 ±1) g. Find the uncertainty in the acceleration a using the **BEST** method only.

Solution:

Let's determine a:

$$\begin{aligned} a &\approx g\frac{M-m}{M+m} \\ &= (9.8)\frac{100-50}{100+50} \\ &\approx 3.3 \end{aligned}$$

Now for the two partial derivatives

$$\frac{\partial a}{\partial M} = g\left[\frac{[1](M+m)-(M-m)[1]}{(M+m)^2}\right] = \frac{2mg}{(M+m)^2}$$

$$\frac{\partial a}{\partial m} = g\left[\frac{[-1](M+m)-(M-m)[1]}{(M+m)^2}\right] = -\frac{2Mg}{(M+m)^2}$$

Therefore putting the derivatives together

$$\begin{aligned} \Delta q &= \sqrt{\left(\frac{\partial a}{\partial M}\Delta M\right)^2 + \left(\frac{\partial a}{\partial m}\Delta m\right)^2} \\ &= \sqrt{\left(\frac{2mg}{(M+m)^2}\cdot\Delta M\right)^2 + \left(-\frac{2Mg}{(M+m)^2}\cdot\Delta m\right)^2} \\ &= \frac{2g}{(M+m)^2}\sqrt{m^2(\Delta M)^2 + M^2(\Delta m)^2} \\ &= \frac{2(9.8)}{(100+50)^2}\sqrt{(50)^2(1)^2 + (100)^2(1)^2} \\ &\approx 0.1 \end{aligned}$$

Finally $a = (3.3 \pm 0.1)\frac{m}{s^2}$